Assets Selection Problem for a Defined Contribution Pension Management under a Market with Inflation

[*]Edikan E. Akpanibah
Department of Mathematics and Statistics,
Federal University Otuoke,
P.M.B 126, Yenagoa,
Bayelsa State, Nigeria.
*edikanakpanibah@gmail.com

Obinichi C. Mandah
Department of Mathematics,
University of Ibadan, Ibadan,
Oyo State, Nigeria.
m.obinichi@gmail.com

Abstract
In this paper, we investigate the optimal strategy for a pension member in a defined contribution pension scheme under a market with inflation and minimum guarantee. We assume the contribution process includes the mandatory contribution and a supplementary contribution to amortize the pension fund which is assumed to be stochastic. Next, the management of the pension considers investments in cash, stock and inflation-linked bond to maximize the expected return of his member at the time of retirement. Using stochastic optimal control method, we derived an optimized problem from the Hamilton Jacobi Bellman (HJB) equation for the value function. Furthermore, we obtain the closed form solution of the optimal strategy for the three assets using constant absolute risk aversion (CARA) utility function and observed that the supplementary contributions has a direct effect on the inflation-linked bond and cash only.

Keyword: *DC Pension scheme, HJB, optimal investment strategy, inflation, supplementary contribution, minimum guarantee.*

1. Introduction

The asset selection problem for a defined contribution pension scheme is a fundamental problem in the area of mathematics of finance and this has triggered a lot of researchers to engage in researching in this area.

Two major types of pension scheme has emerged in the past years in which members of the pension scheme can involved themselves in and this include the defined benefit (DB) and defined contribution (DC). In DB scheme, the employers bear the burden of contribution and the members benefits are defined based on some factors such as age, years of service, level of income etc while in DC scheme, members contribute a specific percentage of their earnings to their pension account and their benefit depend on the expected investment returns. Following the involvement of investment, there is need to understand appropriate ways to invest in order to yield optimal returns and this study has been carried out by many authors some of which include Cairns et al (2006) where they investigated optimal dynamics asset allocation for defined contribution pension scheme by studying the various properties of the optimal asset allocation strategy with and without non-hedge able salary risk and the significance of the alternative optimal strategy pension providers. Gao (2008), studied asset allocation problem under stochastic interest. Blake et al (2012), studied asset allocation problem under a loss-averse preference. Akpanibah and Oghen'oro (2018) studied impact of additional voluntary contribution on the optimal investment strategy in a DC pension scheme with multiple contributors. Also Boulier et al (2001), Xiao et. al (2007), Battocchio and Menoncin (2007), Osu et al (2018), Gerrard et al (2004), Akpanibah and Samaila (2017), Njoku et al (2017) etc studied optimal portfolio problems under various conditions.

The study of optimal investment strategy in a DC pension scheme with minimum guarantee has been studied by some authors, some of which include Nkeki and Nwozu (2012) in their study, their aim was to determine the optimal portfolio values that depend on the minimum guarantee. They observed that a certain fraction of the wealth has to be transferred into the cash account from the indexed bond and stock portfolio to relax the effect of the inflation. In Deelstra et al (2003) the pension manager invests initial wealth and the stochastic contribution into the financial market where the stochastic interest followed the CIR model. Othusitse and Xiaoping (2015) investigate optimal investment problem under inflationary market with minimum guarantee; they introduce a supplementary contribution to amortize the pension scheme and obtain the optimal strategy using constant relative risk aversion (CARA) utility function. In this paper, we modify the work of Othusitse and Xiaoping (2015) by assuming that the supplementary contribution is stochastic unlike in Othusitse and Xiaoping (2015), where the supplementary contribution was deterministic. We will obtain the optimal investment strategy using the constant absolute risk aversion (CARA) utility function.

2. Investment in Financial Market

Let the financial market be complete and frictionless which is continuously open over a fixed time interval $0 \leq t \leq T$, where $T$is the retirement time of a given plan member

Suppose the market is complete and frictionless which is continuously open over a fixed time interval $0 \leq t \leq T$, where $T$is the retirement time of a given plan member and is made up of a risk free asset (cash) and two risky assets(stock and inflation linked bond). Also, consider a complete probability space (Ω, F, P) where Ω is a real space and P is a probability measure, F is the filtration and represent the information generated by the standard two dimensional Brownian motion $\{B_0(t), B_1(t)\}$ whose sources of uncertainties are stock market and inflation rates respectively. The two Brownian motions are related as follows

$$dB_0(t)dB_1(t) = \frac{1}{2}\rho \tag{1}$$

Let $D_c(t)$ denote the price of the risk free asset whose dynamics is given as

$$dD_c(t) = r_R(t)D_c(t)dt \tag{2}$$

Where $r_R(t)$ is the real interest rate generated by the risk free asset and is described

$$dr_R(t) = \big(p_0 - p_1 r_R(t)\big)dt + \sigma_R dB_0 \tag{3}$$

$p_0$is the long term mean level, $p_1$is the rate of mean reversion and σ_R is the instantaneous volatility of the real interest rate

Let $D_s(t)$ denote the price of the risk free asset, whose dynamic is given as

$$dD_s(t) = D_s(r_R(t) + \mu_1\sigma_1 + \mu_2\sigma_2\vartheta)dt + \sigma_1 dB_0 + \sigma_2 dB_1) \tag{4}$$

Where ϑ is the inflation price market risk, μ_1 and μ_2 are the instantaneous risk premiums associated with the positive volatility constants, σ_1 and σ_2 as in Deelstra et al (2000).

Let $D_B(t, I(t))$ denote the price of the inflation indexed bond and the dynamics is given by

$$\frac{dD_B(t,I(t))}{D_B(t,I(t))} = (r_R(t) + \sigma_3\vartheta)dt + \sigma_3 dB_1 \tag{5}$$

Let the salary of the pension member be described by the stochastic differential equation

$$\frac{dL(t)}{L(t)} = \mu_L dt + \sigma_s dB_0 + \sigma_I dB_1 \tag{6}$$

Where μ_L, is the instantaneous rate of the salary, σ_s and σ_I are the volatility of the stock and inflation respectively.
Next, we assume that the contribution process is given by

$$dc(t) = \alpha dt + \beta dB_1 \tag{7}$$

Where α is the mandatory contribution and β is supplementary contribution into the pension account.

3. Minimum Guarantee

The minimum guarantee at any time ($t\epsilon[0, T]$) is represented by the equation below

$$g(t) = \int_0^t h(\tau)c(\tau)e^{i(T-\tau)}d\tau \tag{8}$$

From Nkeki and Nwozo (2013) and Othusitse and Xiaoping (2015), we defined the expected minimum guarantee process as follows
Definition 1
The value of the expected minimum guarantee process is defined as

$$F(t) = E_t[g(t)] = E_t\left[\int_0^t h(\tau)c(\tau)e^{i(T-\tau)}d\tau\right], t \geq 0 \tag{9}$$

Where E_t is the conditional expectation, $h(t)$ is a discounting factor that adjust the real interest rate to the market price risks.
The discounting factor $h(t)$, is given by the SDE

$$dh(t) = h(t)(r_R(t)dt - \pi_1 dB_0 - \pi_2 dB_1) \tag{10}$$

4. Wealth Formulation

Let $W(t)$ denote the wealth of the pension fund at $t \in [0, T]$, let θ_s denote the proportion of the fund invested in stock, $\theta_B$the proportion to be invested in inflation linked bond and$\theta_c = 1 - \theta_S - \theta_B$, the proportion invested in cash. Hence the dynamics of the pension wealth is given by

$$dW(t) = \theta_c W(t)\frac{dD_c(t)}{D_c(t)} + \theta_B(t)W(t)\frac{dD_B(t,I(t))}{D_B(t,I(t))} + \theta_s W(t)\frac{dD_s(t)}{D_s(t)} + dc(t) \tag{11}$$

$$dW(t) = (1 - \theta_S - \theta_B)W(t)\frac{dD_c(t)}{D_c(t)} + \theta_B(t)W(t)\frac{dD_B(t,I(t))}{D_B(t,I(t))} + \theta_s W(t)\frac{dD_s(t)}{D_s(t)} + dc(t) \tag{12}$$

Substituting (2), (4) and (5) into (12) we have

$$dW(t) = (1 - \theta_S - \theta_B)W(t)r_R(t)dt + \theta_B(t)W(t)((r_R(t) + \sigma_3\vartheta)dt + \sigma_3 dB_1) + \theta_s W(t)(r_R(t) + \mu_1\sigma_1 + \mu_2\sigma_2\vartheta)dt + \sigma_1 dB_0 + \sigma_2 dB_1) + \alpha dt + \beta dB_1 \tag{13}$$

$$dW(t) = (W(t)(r_R(t) + \theta_S\mu_1\sigma_1 + \theta_S\mu_2\sigma_2\vartheta + \theta_B\sigma_3\vartheta) + \alpha)dt + W(t)\theta_S\sigma_1 dB_0 + (W(t)(\sigma_2\theta_s + \theta_B\sigma_3) + \beta)dB_1 \tag{14}$$

5. Optimization Problem

In this section we are interested in maximizing the utility of the plan contributor's terminal wealth. Let $H_\pi$represent the utility attained by the plan contributor from a given state w at time t as

$$M_\theta(t,,w) = E_\theta[\,U(W(T) - F_T)\ |, W(t) = w, F(t) = f\,], \tag{15}$$

where$t$ is the time and w is the wealth. Our aim is to obtain the optimal value function

$$M(t,w) = \sup_\theta M_\theta(t,w) \tag{16}$$

and the optimal strategy $\theta = (\theta_c, \theta_B, \theta_{s)}$ such that

$$M_\theta(t,w) = M(t,w). \tag{17}$$

The Hamilton-Jacobi-Bellman (HJB) equation associated with (14) is given below

$$M_t + \sup\Big\{(W(t)(r_R(t) + \theta_S\mu_1\sigma_1 + \theta_S\mu_2\sigma_2\vartheta + \theta_B\sigma_3\vartheta) + \alpha)M_w + \frac{1}{2}\big[\theta_S{}^2\sigma_1{}^2w^2 + \theta_S{}^2\sigma_1{}^2\sigma_2{}^2\rho w^2 + \theta_S\theta_B\sigma_1\sigma_3\rho w^2 + w\beta\theta_S\rho\sigma_1 + \theta_S{}^2\sigma_2{}^2w^2 + 2\theta_S\theta_B\sigma_2\sigma_3w^2 + 2w\beta\theta_S\sigma_2 + \theta_B{}^2\sigma_3{}^2w^2 + 2w\beta\theta_B\sigma_3 + \beta^2\big]M_{ww}\Big\} = 0 \quad (18)$$

To obtain the first order maximizing condition for $\theta_B{}^*$,$\theta_s{}^*$ we differentiate the expression above with respect to θ_S and $\theta_B$respectively and equate it to zero to have

$$\theta_B{}^* = -\frac{1}{\sigma_3}\left(\frac{\vartheta}{w}\frac{M_w}{M_{ww}} + \theta_S\sigma_2 + \frac{1}{2}\theta_S\rho\sigma_1 + \frac{\beta}{w}\right) \quad (19)$$

$$\theta_s{}^* = \left(\frac{\sigma_2\vartheta + \frac{1}{2}\vartheta\rho\sigma_1 - \mu_1\sigma_1 - \mu_2\sigma_2\vartheta}{\sigma_1{}^2(1-\frac{\rho^2}{2})}\right)\frac{M_w}{M_{ww}} \quad (20)$$

Substituting (19) into (18), we have

$$M_t + (r_R(t)w + \alpha - \beta\vartheta)M_w - \frac{1}{2}\vartheta^2\frac{M_w{}^2}{M_{ww}} + \frac{1}{2}w^2\theta_S{}^2\sigma_1{}^2\left(1 - \frac{\rho^2}{2}\right)M_{ww} + \theta_s(\mu_1\sigma_1 + \mu_2\sigma_2\vartheta - \sigma_2\vartheta - \frac{1}{2}\vartheta\rho\sigma_1)wM_w = 0 \quad (21)$$

Substituting (20) into (21) we have

$$M_t + (r_R(t)w + \alpha - \beta\vartheta)M_w + Q\frac{M_w{}^2}{M_{ww}} = 0 \quad (22)$$

Where

$$Q = \frac{1}{\sigma_1{}^2(1-\frac{\rho^2}{2})}\Big\{\vartheta\mu_1\sigma_1\sigma_2 + \mu_2\vartheta^2\sigma_2{}^2 - \vartheta\mu_1\mu_2\sigma_1\sigma_2 - \frac{1}{2}\rho\vartheta^2\sigma_1\sigma_2 + \frac{1}{2}\rho\vartheta^2\sigma_1\sigma_2\mu_2 + \frac{1}{2}\rho\sigma_1{}^2\vartheta\mu_1 - \frac{1}{2}\sigma_2{}^2\vartheta^2 - \frac{1}{2}\sigma_1{}^2\mu_1{}^2 + \sigma_2{}^2\mu_2{}^2\left(\frac{\vartheta^2}{2} - 1\right) + \frac{1}{2}\sigma_1{}^2\vartheta^2\left(\frac{\rho^2}{4} - 1\right)\Big\} \quad (23)$$

$$\theta_s{}^* = \left(\frac{\sigma_2\vartheta + \frac{1}{2}\vartheta\rho\sigma_1 - \mu_1\sigma_1 - \mu_2\sigma_2\vartheta}{\sigma_1{}^2(1-\frac{\rho^2}{2})}\right)\frac{M_w}{M_{ww}} \quad (24)$$

$$\theta_B{}^* = \frac{\left(\left[\frac{1}{2}\vartheta\sigma_1{}^2\rho^2 + 2\mu_2\sigma_2{}^2\vartheta + 2\mu_1\sigma_1\sigma_2 + \mu_2\sigma_1\sigma_2\vartheta\rho + \rho\sigma_1{}^2\mu_1 - 2\vartheta\sigma_2{}^2 - 2\vartheta\sigma_1{}^2 - \sigma_1\sigma_2\vartheta\rho\right]\frac{M_w}{M_{ww}} - \beta\sigma_1{}^2(2-\rho^2)\right)}{w\sigma_1{}^2(2-\rho^2)} \quad (25)$$

Our interest now is to solve (22) for $M(t,w)$ and substitute it into (24) and (25) to obtain the optimal investment strategy for the three assets.

6. Optimal investment strategy for CARA utility function

Assume the pension contributor takes an exponential utility function

$$U(t, W(t)) = M(t,w) \quad (26)$$

$$M(t,w) = -\frac{1}{q}e^{-q(w(t)-f(t))}, \quad q > 0. \quad (27)$$

Where q is the risk averse level

The absolute risk aversion of a decision maker with the utility described in (27) is constant.

$$M_t = -\frac{1}{q}[w_t - f_t]e^{-q(w(t)-f(t))}, M_w = e^{-q(w(t)-f(t))}, \; M_{ww} = -qe^{-q(w(t)-f(t))} \quad (28)$$

Substituting (28) into (22), we have

$$(w_t - f_t) + (r_R(t)w + \alpha - \beta\vartheta) - \frac{Q}{q} = 0 \quad (29)$$

Splitting (29), we have

$$w_t + r_R(t)w = 0 \quad (30)$$

$$f_t - \alpha + \beta\vartheta + \frac{Q}{q} = 0 \quad (31)$$

Solving (30) and (31), we have

$$w(t) = w_0 \exp[\int_0^t r_R(\tau)\, d\tau] \tag{32}$$

$$f(t) = f_0 + \left(\alpha - \beta\vartheta - \frac{Q}{q}\right) t \tag{33}$$

$$M(t,w) = -\frac{1}{q} \exp(-q(w_0 \exp[\int_0^t r_R(\tau)\, d\tau\,] - f_0 + \left(\alpha - \beta\vartheta - \frac{Q}{q}\right) t) \tag{34}$$

Proposition 1

The optimal investment strategy for the three assets is given as

$$\theta_s{}^* = \left(\frac{\sigma_2\vartheta + \frac{1}{2}\vartheta\rho\sigma_1 - \mu_1\sigma_1 - \mu_2\sigma_2\vartheta}{q\sigma_1{}^2(\frac{\rho^2}{2} - 1)}\right) \tag{35}$$

$$\theta_B{}^* = \frac{\left(\left[\frac{1}{2}\vartheta\sigma_1{}^2\rho^2 + 2\mu_2\sigma_2{}^2\vartheta + 2\mu_1\sigma_1\sigma_2 + \mu_2\sigma_1\sigma_2\vartheta\rho + \rho\sigma_1{}^2\mu_1 - 2\vartheta\sigma_2{}^2 - 2\vartheta\sigma_1{}^2 - \sigma_1\sigma_2\vartheta\rho\right]\right)}{qw\sigma_1{}^2(\rho^2 - 2)} - \frac{\beta}{w} \tag{36}$$

$$\theta_c{}^* = 1 - \left(\frac{\sigma_2\vartheta + \frac{1}{2}\vartheta\rho\sigma_1 - \mu_1\sigma_1 - \mu_2\sigma_2\vartheta}{q\sigma_1{}^2\left(\frac{\rho^2}{2} - 1\right)}\right) - \frac{\left(\left[\frac{1}{2}\vartheta\sigma_1{}^2\rho^2 + 2\mu_2\sigma_2{}^2\vartheta + 2\mu_1\sigma_1\sigma_2 + \mu_2\sigma_1\sigma_2\vartheta\rho + \rho\sigma_1{}^2\mu_1 - 2\vartheta\sigma_2{}^2 - 2\vartheta\sigma_1{}^2 - \sigma_1\sigma_2\vartheta\rho\right]\right)}{qw\sigma_1{}^2(\rho^2 - 2)} - \frac{\beta}{w} \tag{37}$$

Proof

Recall, from (24), (25) and (34), we have

$$\theta_s{}^* = \left(\frac{\sigma_2\vartheta + \frac{1}{2}\vartheta\rho\sigma_1 - \mu_1\sigma_1 - \mu_2\sigma_2\vartheta}{\sigma_1{}^2(1 - \frac{\rho^2}{2})}\right)\frac{M_w}{M_{ww}}$$

$$\theta_B{}^* = \frac{\left(\left[\frac{1}{2}\vartheta\sigma_1{}^2\rho^2 + 2\mu_2\sigma_2{}^2\vartheta + 2\mu_1\sigma_1\sigma_2 + \mu_2\sigma_1\sigma_2\vartheta\rho + \rho\sigma_1{}^2\mu_1 - 2\vartheta\sigma_2{}^2 - 2\vartheta\sigma_1{}^2 - \sigma_1\sigma_2\vartheta\rho\right]\frac{M_w}{M_{ww}} - \beta\sigma_1{}^2(2 - \rho^2)\right)}{w\sigma_1{}^2(2 - \rho^2)}$$

$$M(t,w) = -\frac{1}{q} \exp(-q(w_0 \exp[\int_0^t r_R(\tau)\, d\tau\,] - f_0 + \left(\alpha - \beta\vartheta - \frac{Q}{q}\right) t)$$

Differentiating $M(t,w)$ with respect to w, we have

$$M_w = \exp(-q(w_0 \exp[\int_0^t r_R(\tau)\, d\tau\,] - f_0 + \left(\alpha - \beta\vartheta - \frac{Q}{q}\right) t) \tag{38}$$

$$M_{ww} = -q \exp(-q(w_0 \exp[\int_0^t r_R(\tau)\, d\tau\,] - f_0 + \left(\alpha - \beta\vartheta - \frac{Q}{q}\right) t) \tag{39}$$

Substituting (38) and (39) into $\theta_s{}^*$ and $\theta_B{}^*$, we have

$$\theta_s{}^* = \left(\frac{\sigma_2\vartheta + \frac{1}{2}\vartheta\rho\sigma_1 - \mu_1\sigma_1 - \mu_2\sigma_2\vartheta}{q\sigma_1{}^2(\frac{\rho^2}{2} - 1)}\right)$$

$$\theta_B{}^* = \frac{\left(\left[\frac{1}{2}\vartheta\sigma_1{}^2\rho^2 + 2\mu_2\sigma_2{}^2\vartheta + 2\mu_1\sigma_1\sigma_2 + \mu_2\sigma_1\sigma_2\vartheta\rho + \rho\sigma_1{}^2\mu_1 - 2\vartheta\sigma_2{}^2 - 2\vartheta\sigma_1{}^2 - \sigma_1\sigma_2\vartheta\rho\right]\right)}{qw\sigma_1{}^2(\rho^2 - 2)} - \frac{\beta}{w}$$

$$\theta_c{}^* = 1 - \left(\frac{\sigma_2\vartheta + \frac{1}{2}\vartheta\rho\sigma_1 - \mu_1\sigma_1 - \mu_2\sigma_2\vartheta}{q\sigma_1{}^2\left(\frac{\rho^2}{2} - 1\right)}\right) - \frac{\left(\left[\frac{1}{2}\vartheta\sigma_1{}^2\rho^2 + 2\mu_2\sigma_2{}^2\vartheta + 2\mu_1\sigma_1\sigma_2 + \mu_2\sigma_1\sigma_2\vartheta\rho + \rho\sigma_1{}^2\mu_1 - 2\vartheta\sigma_2{}^2 - 2\vartheta\sigma_1{}^2 - \sigma_1\sigma_2\vartheta\rho\right]\right)}{qw\sigma_1{}^2(\rho^2 - 2)} - \frac{\beta}{w}$$

Remark 1

If there is no supplementary contribution, i.e $\beta = 0$, $\theta_B{}^*$ and $\theta_c{}^*$ reduces to

$$\theta_{B_1}{}^* = \frac{\left(\left[\frac{1}{2}\vartheta\sigma_1{}^2\rho^2 + 2\mu_2\sigma_2{}^2\vartheta + 2\mu_1\sigma_1\sigma_2 + \mu_2\sigma_1\sigma_2\vartheta\rho + \rho\sigma_1{}^2\mu_1 - 2\vartheta\sigma_2{}^2 - 2\vartheta\sigma_1{}^2 - \sigma_1\sigma_2\vartheta\rho\right]\right)}{qw\sigma_1{}^2(\rho^2 - 2)}$$

$$\theta_{c_1}{}^{*} = 1 - \left(\frac{\sigma_2\vartheta+\frac{1}{2}\vartheta\rho\sigma_1-\mu_1\sigma_1-\mu_2\sigma_2\vartheta}{q\sigma_1{}^2\left(\frac{\rho^2}{2}-1\right)}\right) - \frac{\left(\left[\frac{1}{2}\vartheta\sigma_1{}^2\rho^2+2\mu_2\sigma_2{}^2\vartheta+2\mu_1\sigma_1\sigma_2+\mu_2\sigma_1\sigma_2\vartheta\rho+\rho\sigma_1{}^2\mu_1-2\vartheta\sigma_2{}^2-2\vartheta\sigma_1{}^2-\sigma_1\sigma_2\vartheta\rho\right]\right)}{qw\sigma_1{}^2(\rho^2-2)}$$

Proposition 2
The optimal strategy for the inflation linked bond when there is no supplementary contribution is greater than when there is supplementary contribution.
Proof
Assume $A = \frac{\left(\left[\frac{1}{2}\vartheta\sigma_1{}^2\rho^2+2\mu_2\sigma_2{}^2\vartheta+2\mu_1\sigma_1\sigma_2+\mu_2\sigma_1\sigma_2\vartheta\rho+\rho\sigma_1{}^2\mu_1-2\vartheta\sigma_2{}^2-2\vartheta\sigma_1{}^2-\sigma_1\sigma_2\vartheta\rho\right]\right)}{qw\sigma_1{}^2(\rho^2-2)} > 0 \; and \; \beta >$ 0
Then
$\theta_B{}^{*} = A - \frac{\beta}{w}and\theta_{B_1}{}^{*} = A$
Since $\beta > 0$, then
$\theta_B{}^{*} = A - \frac{\beta}{w} < A = \theta_{B_1}{}^{*}$
Hence $\theta_B{}^{*} < \theta_{B_1}{}^{*}$
This is also true for $\theta_c{}^{*}$

7. Discussion and Conclusion
7.1 Discussion
From remark 1 and proposition 2, we observed that the supplementary contribution have direct effect on the proportion to be invested in cash and bond but not on stock. Also, when there is no supplementary contribution the fund manager increases its investment in cash and bond.

7.2 Conclusion
We studied the optimal investment strategy in a defined contribution pension scheme under a market affected with inflation and minimum guarantee. We assume the contribution process includes the mandatory contribution and a supplementary contribution to help balance the pension fund which is assumed to be stochastic. Also, the fund manager considers investments in cash, stock and inflation-linked bond to maximize the expected return of his member at the time of retirement. We used stochastic optimal control method to derived an optimized problem. Next, we solved for the explicit solution of the optimal strategy for the three assets using exponential utility function and found out that the supplementary contribution has a direct effect on the inflation-linked bond and cash only.

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